Context-Free Grammars

A Motivating Question

python3

```
>>> (137 + 42) - 2 * 3
173

>>> (60 + 37) + 5 * 8
137

>>> (200 / 2) + 6 / 2
103.0

>>>
```

Mad Libs for Arithmetic

(_____ _____ _____) _____ _____ _____ _____

Int **Op** **Int** **Op** **Int** **Op** **Int**

This only lets us make arithmetic expressions of the form (**Int Op Int**) **Op Int Op Int**.

What about arithmetic expressions that don't follow this pattern?

Slide credit: Amy Liu

Recursive Mad Libs

(int / (int + int))

Expr Op Expr Op Expr

What can an arithmetic expression be?

int	A single number.
Expr Op Expr	Two expressions joined by an operator.
(Expr)	A parenthesized expression.

A ***context-free grammar*** (or ***CFG***) is a recursive set of rules that define a language.

(There's a bunch of specific requirements about what those rules can be; more on that in a bit.)

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`
Expr → **Expr Op Expr**
Expr → `(`**Expr**`)`
Op → `+`
Op → `-`
Op → `×`
Op → `/`

This is called a *production rule*. It says "if you see **Expr**, you can replace it with **Expr Op Expr**."

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

This one says "if you see **Op**, you can replace it with `-`."

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

$$\textbf{Expr} \rightarrow \texttt{int}$$
$$\textbf{Expr} \rightarrow \textbf{Expr Op Expr}$$
$$\textbf{Expr} \rightarrow \texttt{(}\textbf{Expr}\texttt{)}$$
$$\textbf{Op} \rightarrow \texttt{+}$$
$$\textbf{Op} \rightarrow \texttt{-}$$
$$\textbf{Op} \rightarrow \times$$
$$\textbf{Op} \rightarrow \texttt{/}$$

$$
\begin{aligned}
& \textbf{Expr} \\
\Rightarrow\ & \textbf{Expr Op Expr} \\
\Rightarrow\ & \textbf{Expr Op } \texttt{int} \\
\Rightarrow\ & \texttt{int } \textbf{Op } \texttt{int} \\
\Rightarrow\ & \texttt{int / int}
\end{aligned}
$$

These red symbols are called ***nonterminals***. They're placeholders that get expanded later on.

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Expr

⇒ **Expr Op Expr**

⇒ **Expr Op** `int`

⇒ `int` **Op** `int`

⇒ `int / int`

The symbols in blue monospace are **terminals**. They're the final characters used in the string and never get replaced.

Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - a set of ***nonterminal symbols*** (also called ***variables***),
  - a set of ***terminal symbols*** (the ***alphabet*** of the CFG),
  - a set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - a ***start symbol*** (which must be a nonterminal) that begins the derivation. By convention, the start symbol is the one on the left-hand side of the first production.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Some CFG Notation

- In today's slides, capital letters in **Bold Red Uppercase** will represent nonterminals.
  - e.g. **A**, **B**, **C**, **D**
- Lowercase letters in **`blue monospace`** will represent terminals.
  - e.g. **`t`**, **`u`**, **`v`**, **`w`**
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - e.g. $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$
- You don't need to use these conventions on your own; just make sure whatever you do is readable.

A Notational Shorthand

Expr → `int` | **Expr Op Expr** | `(`**Expr**`)`

Op → `+` | `-` | `×` | `/`

Derivations

Expr
⇒ **Expr Op Expr**
⇒ **Expr Op** `(`**Expr**`)`
⇒ **Expr Op** `(`**Expr Op Expr**`)`
⇒ **Expr** `×` `(`**Expr Op Expr**`)`
⇒ `int ×` `(`**Expr Op Expr**`)`
⇒ `int × (int` **Op Expr**`)`
⇒ `int × (int` **Op** `int)`
⇒ `int × (int + int)`

- A sequence of zero or more steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string α derives string ω, we write $\alpha \Rightarrow^* \omega$.
- In the example on the left, we see that

  Expr ⇒* `int × (int + int)`.

Expr → `int` | **Expr Op Expr** | `(`**Expr**`)`
Op → `+` | `-` | `×` | `/`

The Language of a Grammar

- If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega} \}$$

- That is, $\mathscr{L}(G)$ is the set of strings of terminals derivable from the start symbol.

If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \omega \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \omega \}$$

Consider the following CFG G over $\Sigma = \{$**a**, **b**, **c**, **d**$\}$:

$$\mathbf{Q} \to \mathbf{Qa} \mid \mathbf{dH}$$
$$\mathbf{H} \to \mathbf{bHb} \mid \mathbf{c}$$

Which of the following strings are in $\mathscr{L}(G)$?

dca
dc
cad
bcb
dHaa

Answer at ***https://cs103.stanford.edu/pollev***

Context-Free Languages

- A language L is called a ***context-free language*** (or CFL) if there is a CFG G such that $L = \mathscr{L}(G)$.
- Questions:
  - How are context-free and regular languages related?
  - How do we design context-free grammars for context-free languages?

CFGs and Regular Expressions

- CFGs consist purely of production rules of the form **A** → ***ω***. They do not have the regular expression operators * or ∪.
- You can use the symbols * and ∪ if you'd like in a CFG, but they just stand for themselves.
- Consider this CFG G:

$$\mathbf{S} \rightarrow \texttt{a*b}$$

- Here, $\mathscr{L}(G) = \{\texttt{a*b}\}$ and has cardinality one. That is, $\mathscr{L}(G) \neq \{\ \texttt{a}^n\texttt{b} \mid n \in \mathbb{N}\ \}$.

CFGs and Regular Expressions

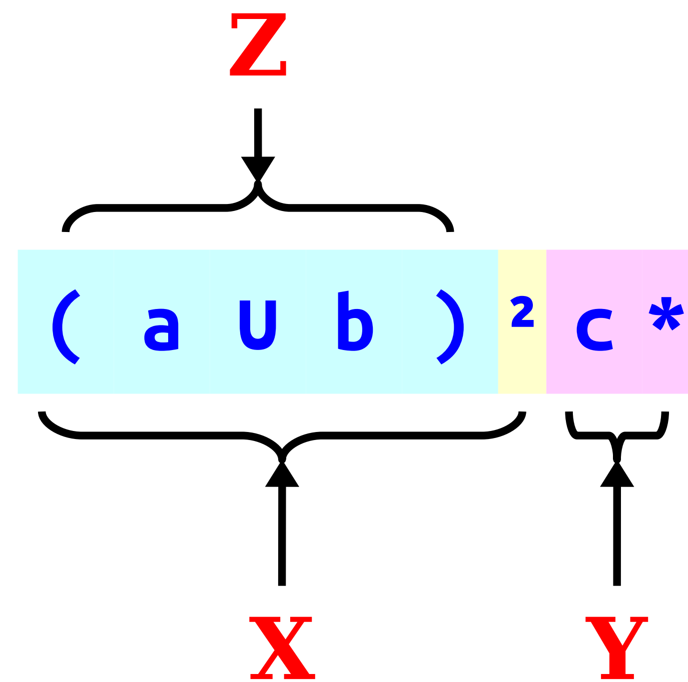
- ***Theorem:*** Every regular language is context-free.
- ***Proof idea:*** Show how to convert an arbitrary regular expression into a context-free grammar.

$S \rightarrow XY$

$X \rightarrow ZZ$

$Z \rightarrow$ a | b

$Y \rightarrow$ cY | ε



The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

a	a	a	a	b	b	b	b

$$\mathscr{L}(G) = \{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$$


Regular Languages
CFLs
All Languages

Why the Extra Power?

- Why do CFGs have more power than regular expressions?
- ***Intuition:*** Derivations of strings have unbounded "memory."

$$S \to \mathtt{a}S\mathtt{b} \mid \varepsilon$$

a	a	a	a	b	b	b	b

Time-Out for Announcements!

Problem Set Seven

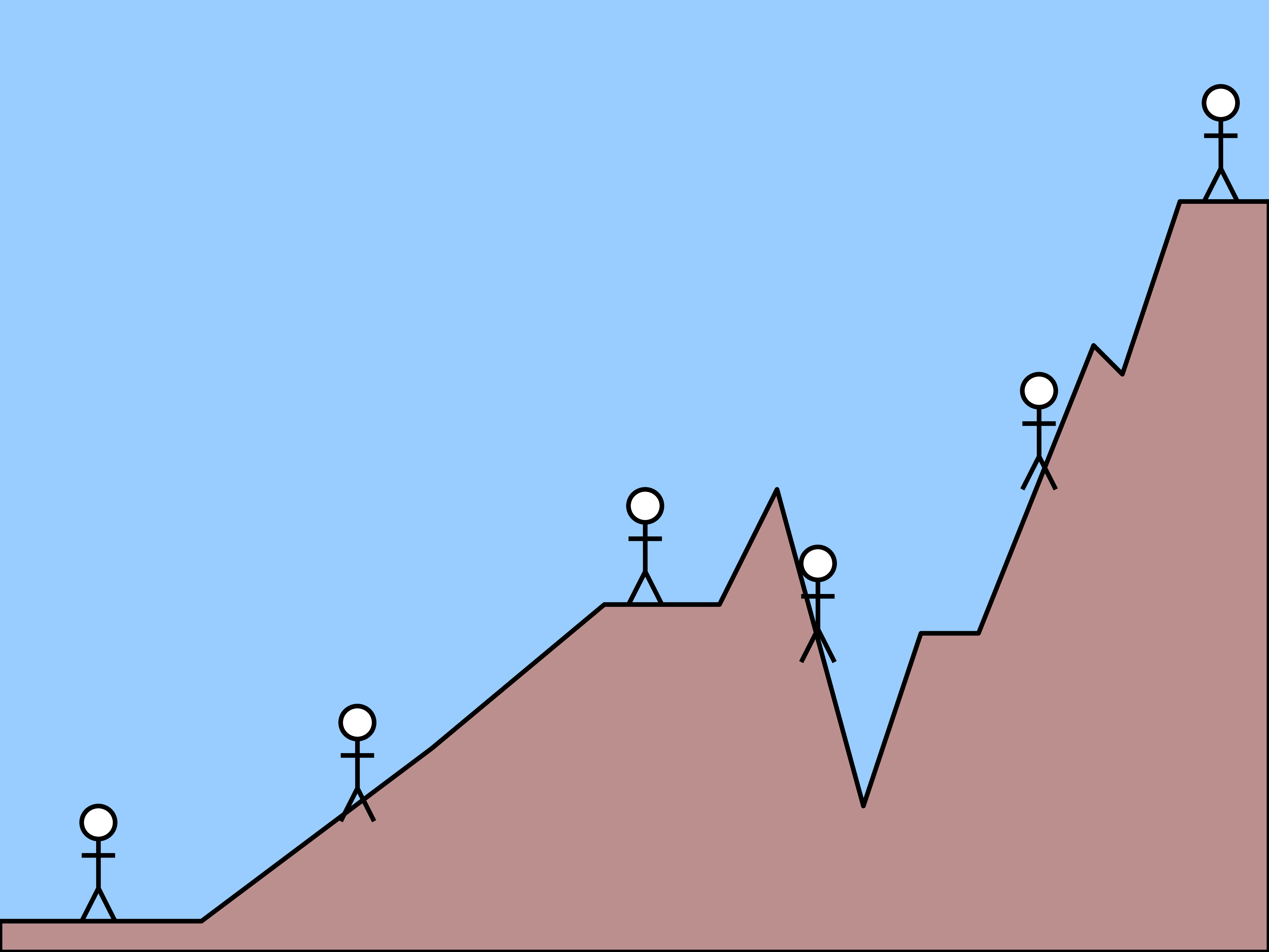
- Problem Set Six was due today at 1:00PM.
  - You can extend the deadline to Saturday at 1:00PM using a late day.
- Problem Set Seven goes out today. It's due next Friday at 1:00PM.
  - It's all about regular expressions, properties of regular languages, and gives a first glimpse at nonregular languages.
  - We've tuned the length given that you have a midterm next Monday.

Second Midterm Logistics

- Our second midterm exam is next ***Monday, November 11th*** from ***7PM – 10PM***. Check the website for your seating assignment; note that it has changed since the first midterm.
- Topic coverage is primarily lectures 06 – 13 (functions through induction) and PS3 – PS5. Finite automata and onward won't be tested here.
  - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- The exam is closed-book and closed-computer. You can bring one double-sided 8.5" × 11" sheet of notes with you.
- Students with OAE accommodations: you should have heard from us with alternate exam locations. If you haven't, contact us ASAP.

Our Advice

- ***Stay fed and rested.*** You are not a brain in a jar. You are a rich, complex, beautiful human being. Please take care of yourself.
- ***Read all questions before diving into them.*** You don't have to go sequentially. Read over each problem so you know what to expect, then pick whichever one looks easiest and start there.
- ***Reflect on how far you've come.*** How many of these questions would you have been able to *understand* two months ago? That's the mark that you're learning something!



Three Questions

- What’s something you know now that, at the start of the quarter, you knew you didn’t know?
- What’s something you know now that, at the start of the quarter, you *didn’t* know you didn’t know?
- What’s something you *don’t* know now that, at the start of the quarter, you *didn’t* know you didn’t know?

Back to CS103!

Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.
- Check our online “Guide to CFGs” for more information about CFG design.
- We’ll hit the highlights in the rest of this lecture.

Designing CFGs

- Let $\Sigma = \{\texttt{a}, \texttt{b}\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for L by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If ω is a palindrome, then **a**ω**a** and **b**ω**b** are palindromes.
  - No other strings are palindromes.

$$\mathbf{S} \rightarrow \varepsilon \mid \texttt{a} \mid \texttt{b} \mid \texttt{a}\mathbf{S}\texttt{a} \mid \texttt{b}\mathbf{S}\texttt{b}$$

Designing CFGs

- Let $\Sigma = \{ \texttt{\{}, \texttt{\}} \}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a string of balanced braces} \}$
- Some sample strings in L:

```
{{{}}}
{{}}{}
{{}{}}{{}{}}
{{{{{}}}{{}}}}
ε
{}{}
```

Designing CFGs

- Let Σ = {{, }} and let L = {w ∈ Σ* | w is a string of balanced braces }
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace.

{{{}{{}}}{{}}} | {{}}{{{}}}

Designing CFGs

- Let Σ = { **{**, **}** } and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced braces $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace. Removing the first brace and the matching brace forms two new strings of balanced braces.

S → {S}S | ε

Designing CFGs

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{ w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$

Which of these CFGs have language L?

S → aSb | bSa | ε

S → abS | baS | ε

S → abSba | baSab | ε

S → SbaS | SabS | ε

Answer at
https://cs103.stanford.edu/pollev

Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!
- You'll design your own CFG for this language on Problem Set 8.

Designing CFGs

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{\mathtt{a}, \stackrel{?}{=}\}$ and let $L = \{\mathtt{a}^n \stackrel{?}{=} \mathtt{a}^n \mid n \in \mathbb{N}\}$.
- Is the following a CFG for L?

$$\mathbf{S} \rightarrow \mathbf{X} \stackrel{?}{=} \mathbf{X}$$

$$\mathbf{X} \rightarrow \mathtt{a}\mathbf{X} \mid \varepsilon$$

$$
\begin{aligned}
& \mathbf{S} \\
\Rightarrow\ & \mathbf{X} \stackrel{?}{=} \mathbf{X} \\
\Rightarrow\ & \mathtt{a}\mathbf{X} \stackrel{?}{=} \mathbf{X} \\
\Rightarrow\ & \mathtt{aa}\mathbf{X} \stackrel{?}{=} \mathbf{X} \\
\Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathbf{X} \\
\Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathtt{a}\mathbf{X} \\
\Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathtt{a}
\end{aligned}
$$

Finding a Build Order

- Let $\Sigma = \{$a, ≟$\}$ and let $L = \{$ aⁿ≟aⁿ $\mid n \in \mathbb{N}\}$.
- To build a CFG for L, we need to be more clever with how we construct the string.
  - If we build the strings of a's independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of a's at the same time.
- Here's one possible grammar based on that idea:

S → ≟ | aSa

S
⇒ aSa
⇒ aaSaa
⇒ aaaSaaa
⇒ aaa≟aaa

Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

Applications of Context-Free Grammars

CFGs for Programming Languages

$$
\begin{aligned}
\textbf{BLOCK} &\rightarrow \textbf{STMT} \\
&\mid\ \texttt{\{}\ \textbf{STMTS}\ \texttt{\}} \\
\textbf{STMTS} &\rightarrow \varepsilon \\
&\mid\ \textbf{STMT STMTS} \\
\textbf{STMT} &\rightarrow \textbf{EXPR}\texttt{;} \\
&\mid\ \texttt{if (}\textbf{EXPR}\texttt{)}\ \textbf{BLOCK} \\
&\mid\ \texttt{while (}\textbf{EXPR}\texttt{)}\ \textbf{BLOCK} \\
&\mid\ \texttt{do}\ \textbf{BLOCK}\ \texttt{while (}\textbf{EXPR}\texttt{);} \\
&\mid\ \textbf{BLOCK} \\
&\mid\ \ldots \\
\textbf{EXPR} &\rightarrow \texttt{identifier} \\
&\mid\ \texttt{constant} \\
&\mid\ \textbf{EXPR}\ \texttt{+}\ \textbf{EXPR} \\
&\mid\ \textbf{EXPR}\ \texttt{-}\ \textbf{EXPR} \\
&\mid\ \textbf{EXPR}\ \texttt{*}\ \textbf{EXPR} \\
&\mid\ \ldots
\end{aligned}
$$

Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program "means."
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? ***Take CS143!***

Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- The **Stanford Parser** project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

Next Time

- ***No class Monday.***
- ***Then, when we get back…***
    - ***Turing Machines***
        - What does a computer with unbounded memory look like?
        - How would you program it?